Sketch of solution to Homework 3

Q1 Let $x = \inf X$, $y = \inf Y$. For any $p \in Y$, $p \geq y$. Since $Y \supset X$, $q \geq y$ for any $q \in X$ as well. Hence, $x \geq y$.

For any set of open intervals such that $B \subset \bigcup I_i$, $A \subset \bigcup I_i$. Hence, by the ordering of inf

$$m^*(A) \leq m^*(B).$$

Q2 Suppose $B_n \in \mathfrak{A}$ for all $n \in \mathbb{N}$. Consider $A_n = B_n \setminus \cup_{i=1}^{n-1} B_i$. Then A_n are all disjoint. By assumption, $\cup B_n = \cup A_n \in \mathfrak{A}$. The only if part is trivial.

Q3 We prove it by induction on the number n of the intervals. When $n = 1$, it is trivial. Suppose it is true for $n = k$. If now $[a, b] \subset \cup_{i=1}^{k+1} I_i$, we may assume $a \in I_1 = (x_1, y_1)$, $y_1 < b$. Then $[y_1, b]$ is covered by $I_2, ..., I_{k+1}$. By induction hypothesis,

$$b - y_1 \leq \sum_{i=2}^{k+1} |I_i|.$$

Hence,

$$b - a \leq y_1 - a + \sum_{i=2}^{k+1} |I_i| \leq \sum_{i=1}^{k+1} |I_i|.$$